

Wellposedness of a swimming model in the 3- D incompressible fluid governed by the nonstationary Stokes equation.

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Abstract

We introduce and investigate the wellposedness of a model describing the self-propelled motion of a *small* abstract *swimmer* in the 3- D incompressible fluid governed by the nonstationary Stokes equation, typically associated with the low Reynolds numbers. It is assumed that the swimmer's body consists of finitely many subsequently connected parts, identified with the fluid they occupy, linked by rotational and elastic Hooke's forces. In this paper we are attempting to extend the 2- D version of this model, introduced in [18]-[20], to the 3- D case. Models like this are of interest in biological and engineering applications dealing with the study and design of propulsion systems in fluids.

Key words: Swimming models, hybrid systems, nonstationary Stokes fluid.

1. Introduction and problem formulation. It seems that the first quantitative research in the area of swimming phenomenon was aimed at the biomechanics of specific biological species: Gray [10](1932), Gray and Hancock [11] (1951), Taylor [36] (1951), [37] (1952), Wu [41] (1971), Lighthill [25] (1975), and others. These efforts resulted in the derivation of a number of mathematical models (linked the size of Reynolds number) for swimming motion in the whole R^2 - or R^3 -spaces with the swimmer to be used as the reference frame, see, e.g., Childress [3] (1981) and the references therein. Such approach however requires some modification if one wants to track the actual position of swimmer in a fluid.

A different modeling approach was proposed by Peskin in the computational mathematical biology (see Peskin [30] (1975), Fauci and Peskin [6] (1988), Fauci [7] (1993), Peskin and McQueen [31] (1994) and the references therein), where a swimmer is modeled as an immaterial *immersed boundary* identified with the fluid, further discretized for computational purposes on some grid. In this case a fluid equation is to be complemented by a coupled *infinite* dimensional differential equation for the aforementioned "immersed boundary".

In this paper we intend to deal with the swimming phenomenon in the framework of *non-stationary* PDE's along the *immersed body* approach summarized in Khapalov [20] (2005-2010). Namely, in [18] (2005), inspired by the ideas of the above-cited Peskin's method, we introduced a 2- D model for "*small*" *flexible swimmers* assuming that their bodies are identified with the fluid occupying their shapes. This approach views such a swimmer as the already *discretized* aforementioned immersed boundary supported on the respective grid cells, see, e.g., Fig. 1 below. Our model offered two novel features: (1) it was set in a *bounded* domain with (2) governing equations to be a fluid equation coupled with a *system of ODE's* describing the spatial position of swimmer within the space domain. We established the wellposedness of this model up to the contact of a swimmer at hand either with the boundary of space domain or with itself. The need of such type of models was motivated by the intention to investigate controllability properties of swimming phenomenon (see [20]).

Our goal in this paper is to investigate the wellposedness of a 3- D version of this model.

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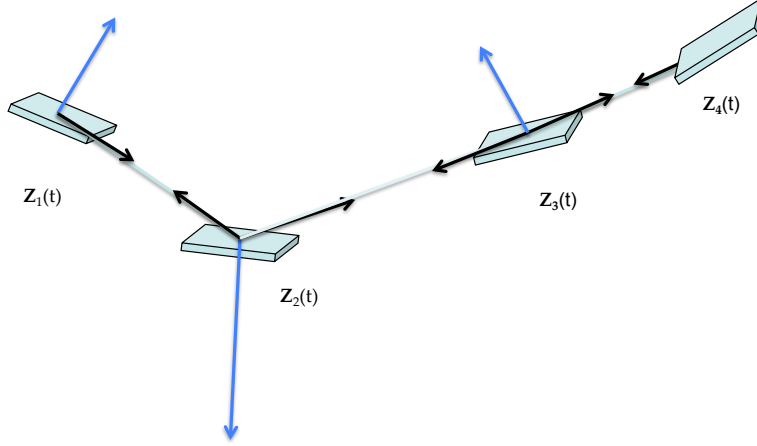


Figure 1: 4-parallelepipeds swimmer with all elastic forces and rotation forces about $z_2(t)$ only.

Further remarks on bibliography. It should be noted that, the classical mathematical issue of wellposedness of a swimming model as a system of PDE's for the first time was apparently addressed by Galdi [8] (1999) for a model of swimming micromotions in R^3 (with the swimmer serving as the reference frame).

Another available approach to modeling of swimming motion (apparently, initiated by the work Shapere and Wilczek [34] (1989)) exploits the idea that the swimmer's body shape transformations during the actual swimming process can be viewed as a set-valued map in time. The respective models describe swimmer's position via such maps, see [13] (1981), [33] (2008), [4] (2011) and the references therein. Some models treat these maps as a priori prescribed, in which case the crux of the problem is to identify which maps are admissible, i.e., compatible with the principle of self-propulsion of swimming locomotion. In the case when the aforementioned motion map is not a priori prescribed (i.e., it will be defined at each moment of time by swimmer's internal forces and the interaction of its body with the resisting surrounding medium), the model will have to include extra equations, see, e.g., [40] in the framework of the immersed boundary method and the references therein.

More recently, a number of significant efforts, both theoretical and experimental, were made to study models of possible bio-mimetic mechanical devices which employ the change of their geometry, inflicted by internal forces, as the means for self-propulsion, see, e.g., S. Hirose [16] (1993), Mason and Burdick [26] (2000); McIsaac and Ostrowski [27] (2000); Martinez and J. Cortes [28] (2001); Trintafyllou et al. [39] (2000); Morgansen et al. [29] (2001); Fakuda et al. [5] (2002); Guo et al. [12] (2002); Hawthorne et al. [14] (2004), and the references therein. It was also recognized that

sophistication and complexity of design of bio-mimetic robots give rise to control-theoretic methods, see, e.g., Koiller et al. [23] (1996); McIsaac and Ostrowski [27] (2000); Martinez and Cortes [28] (2001); Trintafyllou et al. [39] (2000); San Martin et al. [32] (2007), Alouges et al. [1] (2008), Sigalotti and Vivalda [35] (2009), and the references therein. It should be noted however that the above-cited results deal with control problems in the framework of ODE's only.

A number of attempts were made along these lines to introduce various reduction techniques to convert swimming model equations into systems of ODE's, namely, by making use of applicable analytical considerations, empiric observations and experimental data, see, e.g., Becker et al [2] (2003); Kanso et al. [17] (2005); San Martin et al. [32] (2007); Alogues et al. [1] (2008), and the references therein.

Problem formulation for 3-D swimming model. We consider the following model, consisting of two *coupled* systems of equations: one is a PDE system– for the fluid, governed by the *nonstationary 3-D Stokes* equation, and the other is an ODE system– for the *position of the swimming object (or swimmer)* in it:

$$\begin{aligned} \frac{\partial y}{\partial t} &= \nu \Delta y + F(z, v) - \nabla p \quad \text{in } Q_T = \Omega \times (0, T), \quad y = (y_1, y_2, y_3) \\ \operatorname{div} y &= 0 \quad \text{in } Q_T, \quad y = 0 \quad \text{in } \Sigma_T = \partial\Omega \times (0, T), \quad y|_{t=0} = y_0 \quad \text{in } \Omega, \end{aligned} \quad (1.1)$$

$$\frac{dz_i}{dt} = \frac{1}{\operatorname{mes}\{S_i(0)\}} \int_{S_i(z_i(t))} y(x, t) dx, \quad z_i(0) = z_{i0}, \quad i = 1, \dots, n, \quad n > 2, \quad (1.2)$$

where for $t \in [0, T]$:

$$\begin{aligned} z(t) &= (z_1(t), \dots, z_n(t)), \quad z_i(t) \in R^3, \quad i = 1, \dots, n, \quad v(t) = (v_1(t), \dots, v_{n-2}(t)) \in R^{n-2}, \\ F(z, v) &= \sum_{i=2}^n [\xi_{i-1}(x, t) k_{i-1} \frac{(\|z_i(t) - z_{i-1}(t)\|_{R^3} - l_{i-1})}{\|z_i(t) - z_{i-1}(t)\|_{R^3}} (z_i(t) - z_{i-1}(t)) \\ &\quad + \xi_i(x, t) k_{i-1} \frac{(\|z_i(t) - z_{i-1}(t)\|_{R^3} - l_{i-1})}{\|z_i(t) - z_{i-1}(t)\|_{R^3}} (z_{i-1}(t) - z_i(t))] \\ &+ \sum_{i=2}^{n-1} v_{i-1}(t) \left[\xi_{i-1}(x, t) (A_i(z_{i-1}(t) - z_i(t)))(t) - \xi_{i+1}(x, t) \frac{\|z_{i-1}(t) - z_i(t)\|_{R^3}^2}{\|z_{i+1}(t) - z_i(t)\|_{R^3}^2} (B_i(z_{i+1}(t) - z_i(t)))(t) \right] \\ &- \sum_{i=2}^{n-1} \xi_i(x, t) v_{i-1}(t) \left[(A_i(z_{i-1}(t) - z_i(t)))(t) - \frac{\|z_{i-1}(t) - z_i(t)\|_{R^3}^2}{\|z_{i+1}(t) - z_i(t)\|_{R^3}^2} (B_i(z_{i+1}(t) - z_i(t)))(t) \right]. \end{aligned} \quad (1.3)$$

In the above, Ω is a bounded domain in R^3 with boundary $\partial\Omega$ of class C^2 , $y(x, t)$ and $p(x, t)$ are respectively the velocity and the pressure of the fluid at point $x = (x_1, x_2, x_3) \in \Omega$ at time t , while ν is a kinematic viscosity constant. Let us explain the terms in (1.1)-(1.3) in more detail.

Swimmer: The swimmer in (1.1)-(1.3) is modeled as a collection of n bounded sets $S_i(z_i(t))$, $i = 1, \dots, n$ of non-zero measure (such as balls, parallelepipeds, etc.), identified with the fluid within the space which they occupy. These sets are assumed to be open bounded connected sets symmetric relative to the points $z_i(t)$'s which are their centers of mass. The sets $S_i(z_i(t))$'s are viewed as the

given sets $S_i(0)$'s ("0" stands of the origin) that have been shifted to the respective positions $z_i(t)$'s without changing their orientation in space. Respectively,

$$\xi_i(x, t) = \begin{cases} 1, & \text{if } x \in S_i(z_i(t)), \\ 0, & \text{if } x \in \Omega \setminus S_i(z_i(t)), \end{cases} \quad i = 1, \dots, n. \quad (1.4)$$

Throughout the paper we assume that each $S_i(0)$ lies in a "small" neighborhood of the origin of given radius $r > 0$, while $S_i(a)$ denotes the set $S_i(0)$ shifted to point a . Denote by

$$S^0 = \max_{i=1, \dots, n} \{\text{mes} \{S_i(0)\}\}, \quad S_0 = \min_{i=1, \dots, n} \{\text{mes} \{S_i(0)\}\}.$$

Forces: We assume that these sets are subsequently linked by forces described by the term $F(z, v)$. No "actual" physical links between sets $S_i(z_i(t))$ are assumed (i.e., they are assumed to be negligible in terms of affect on the resisting surrounding fluid). The forces in (1.3) are *internal*, relative to the swimmer – their sum is zero. We assume that a force applied to a set $S(z_i(t))$ acts evenly upon all its points, and, as such, it creates an external force on the fluid surrounding $S(z_i(t))$.

The structural integrity of the swimming object is preserved by the elastic forces acting according to Hooke's Law. They act along the lines connecting the respective adjacent centers $z_i(t)$'s when the distances between any two adjacent points $z_{i-1}(t)$ and $z_i(t)$, $i = 2, \dots, n$ deviate from the respective given values $l_{i-1} > 0$, $i = 2, \dots, n$ as described in the first sum in (1.3). The parameters $k_i > 0$, $i = 1, \dots, n-1$ characterize the rigidity of the links $z_{i-1}(t)z_i(t)$, $i = 2, \dots, n$. The matching pairs of these forces between $z_{i-1}(t)$ and $z_i(t)$, and between $z_i(t)$ and $z_{i+1}(t)$ are shown on Fig. 1.

The 2nd sum in (1.3) describes the rotation forces about any on the points $z_i(t)$, $i = 2, \dots, n-1$ which make the adjacent points to rotate about it perpendicular to the lines connecting the respective $z_i(t)$'s. To satisfy the 3rd Newton's Law, these forces lie in the same plane along with the matching counter-force given in the 3rd sum in (1.3). Respectively,

$$A_i = A_i(z_{i-1}(t), z_i(t), z_{i+1}(t)) : R^3 \rightarrow \text{span} \{z_{i-1}(t) - z_i(t), z_{i+1}(t) - z_i(t)\},$$

$$B_i = B_i(z_{i-1}(t), z_i(t), z_{i+1}(t)) : R^3 \rightarrow \text{span} \{z_{i-1}(t) - z_i(t), z_{i+1}(t) - z_i(t)\}$$

denote a *nonlinear* mappings, *defined at each moment of time by three vectors* $z_{i-1}(t), z_i(t), z_{i+1}(t)$, such that for $i = 2, \dots, n-1$:

•

$$(z_{i-1}(t) - z_i(t))'[(A_i(z_{i-1}(t) - z_i(t)))(t)] = 0, \quad (z_{i+1}(t) - z_i(t))'[(B_i(z_{i+1}(t) - z_i(t)))(t)] = 0;$$

•

$$\begin{aligned} \|(A_i(z_{i-1}(t) - z_i(t)))(t)\|_{R^3} &= \|z_{i-1}(t) - z_i(t)\|_{R^3}, \\ \|(B_i(z_{i+1}(t) - z_i(t)))(t)\|_{R^3} &= \|z_{i+1}(t) - z_i(t)\|_{R^3}; \end{aligned}$$

- and the directions of vectors $(A_i(z_{i-1}(t) - z_i(t)))(t)$ and $(B_i(z_{i+1}(t) - z_i(t)))(t)$ are such that they correspond to either folding or unfolding motion of lines $z_{i-1}(t)z_i(t)$ and $z_i(t)z_{i+1}(t)$ relative to the point $z_i(t)$.

The magnitudes and directions of the rotation forces are determined by the given coefficients $v_i(t)$, $i = 1, \dots, n-2$. The choice of fractional coefficients at terms $A_i(z_{i+1}(t) - z_i(t))$ in (1.3) ensures that the momentum of swimmer's internal forces is conserved at any $t \in (0, T)$ (see calculations in [19] in the 2-D case). A matching pair of rotation forces, generated by $z_2(t)$ for the adjacent points, is shown on Fig. 1.

Swimmer's motion. Dynamics of points $z_i(t)\xi_i(x,t), i = 1, \dots, n$ are determined by the average motion of the fluid within their respective supports $S_i(z_i(t))$'s as described in (1.2).

Local and global approach to solutions of (1.1)-(1.3). Note that, when the adjacent points in the swimmer's body share the same position in space, the forcing term F in (1.3) and hence model (1.1)-(1.3) become undefined. While such situation *mathematically* seems possible, it does not have to happen. First of all, one can address the issue of local existence of solutions to (1.1)-(1.3) on some "small" time-interval $(0, T)$, assuming that initially model (1.1)-(1.3) is well-defined in the above sense. This is the primary subject of this paper (see the next section). Then the question of global existence can be viewed as the issue of suitable selection of coefficients v_i 's with the purpose to ensure that the aforementioned ill-posed situation is avoided.

In model (1.1)-(1.3) we chose the fluid governed by the *nonstationary* Stokes equation which, along with its stationary version, is a typical choice of fluid for micro-swimmers (the case of Low Reynolds numbers). The empiric reasoning behind this is that, due to the small size of swimmer, the inertia terms in the Navier-Stokes equation, containing the 1-st order derivatives in t and x , can be omitted, provided that the frequency parameter of the swimmer at hand is a quantity of order unity. However, it was noted that a microswimmer (e.g., a nano-size robot) may use a rather high frequency of motion, which may justify at least in some cases the need for the term y_t in Stokes model equations. In general, it seems reasonable to suggest that the presence of this term (in a number of cases) can provide a better approximation of the Navier-Stokes equation than the lack of it. We also point out that in [6], [7], [31] the full-size Navier-Stokes equation is used for micro-swimmers. It also seems that the methods we use for the nonstationary Stokes equation (as opposed to stationary Stokes equation), may serve as a natural step toward the swimming models, based on the Navier-Stokes equation.

2. Main result: Local existence and uniqueness. Let $\dot{J}(\Omega)$ denote the set of infinitely differentiable vector functions with values in R^3 which have compact support in Ω and are divergence-free, i.e., $\text{div}\phi = 0$ in Ω . Denote by $J_o(\Omega)$ the closure of this set in the $(L^2(\Omega))^3$ -norm and by $G(\Omega)$ denote the orthogonal complement of $J_o(\Omega)$ in $(L^2(\Omega))^3$ (see, e.g., [24], [38]). In $\dot{J}(\Omega)$ introduce the scalar product

$$[\phi_1, \phi_2] = \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \phi_{1j_{x_i}} \phi_{2j_{x_i}} dx, \quad \phi_1(x) = (\phi_{11}, \phi_{12}, \phi_{13}), \quad \phi_2(x) = (\phi_{21}, \phi_{22}, \phi_{23}).$$

Denote by $H(\Omega)$ the Hilbert space which is the completion of $\dot{J}(\Omega)$ in the norm

$$\|\phi_1\|_{H(\Omega)} = \sqrt{\int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \phi_{1j_{x_i}}^2 dx}.$$

Everywhere below we will assume the following two assumptions:

Assumption 2.1. For the given $r > 0$, defining the size of sets $S_i(0)$ in (1.4), assume that

$$l_{i-1} > 2r, \quad i = 2, \dots, n; \quad \overline{S_i}(z_i(0)) \subset \Omega, \quad \|z_{i,0} - z_{j,0}\|_{R^3} > 2r, \quad i, j = 1, \dots, n, \quad i \neq j; \quad (2.1)$$

and the sets $S_i(0), i = 1, \dots, n$ are such that

$$\int_{(S_i(0) \cup S_i(h)) \setminus (S_i(0) \cap S_i(h))} dx = \int_{\Omega} |\xi_i(x) - \xi_i(x-h)| dx \leq C \|h\|_{R^3} \quad \forall h \in B_{h_0}(0) \quad (2.2)$$

for some positive constants h_0 and C , where $\xi_i(x)$ is the characteristic function of $S_i(0)$ and $B_{h_0}(0) = \{x \mid \|x\|_{R^3} < h_0\} \subset R^3$.

Conditions (2.1) mean that at time $t = 0$, any two sets $S_i(z_i(0))$ do not overlap, and that the swimmer lies in Ω . Condition (2.2) is a regularity assumption of Lipschitz type regarding the shift of the set $S(0)$. It is satisfied, for instance, for balls and parallelepipeds.

Assumption 2.2. Assume that within some $(R^3)^n$ neighborhood $\mathbf{G}(z(0)) \subset (R^3)^n$ of the initial datum in (1.2) the mappings A_i and B_i are Lipschitz for all $i = 2, \dots, n-1$ in the following sense:

$$\begin{aligned} & \| A_i(a_{i-1}, a_i, a_{i+1})(a_{i-1} - a_i) - A_i(b_{i-1}, b_i, b_{i+1})(b_{i-1} - b_i) \|_{R^3} \\ & \leq L \{ \| a_{i-1} - b_{i-1} \|_{R^3} + \| a_i - b_i \|_{R^3} + \| a_{i+1} - b_{i+1} \|_{R^3} \}, \\ & \| B_i(a_{i-1}, a_i, a_{i+1})(a_{i+1} - a_i) - B_i(b_{i-1}, b_i, b_{i+1})(b_{i+1} - b_i) \|_{R^3} \\ & \leq L \{ \| a_{i-1} - b_{i-1} \|_{R^3} + \| a_i - b_i \|_{R^3} + \| a_{i+1} - b_{i+1} \|_{R^3} \}, \end{aligned}$$

for any $a_{i\pm 1}, a_i, b_{i\pm 1}, b_i \in \mathbf{G}(z(0))$, where $L > 0$ is a constant and A_i 's and B_i 's are defined as in the introduction by three respective vectors a_{i-1}, a_i, a_{i+1} .

Assumption 2.2 can be satisfied if, e.g., the points $a_{i-1} = z_{i-1}(0), a_i = z_i(0), a_{i+1} = z_{i+1}(0), i = 2, \dots, n-1$ are not on the same line and the mappings A_i and B_i are selected by making use of Gramm-Schmidt orthogonalization procedure for vectors $a_{i-1} - a_i$ and $a_{i+1} - a_i$. Alternatively, we can define A_i 's and B_i 's, making use of the cross-product:

$$\begin{aligned} (A_i(z_{i-1}(t) - z_i(t))(t) &= e_1(t) \| z_{i-1}(t) - z_i(t) \|_{R^3}, \\ (B_i(z_{i+1}(t) - z_i(t))(t) &= e_2(t) \| z_{i+1}(t) - z_i(t) \|_{R^3}, \quad e_i(t) = \frac{v_i(t)}{\|v_i(t)\|_{R^3}}, \quad i = 1, 2, \\ v_1(t) &= (z_{i-1}(t) - z_i(t)) \times [(z_{i-1}(t) - z_i(t)) \times (z_{i+1}(t) - z_i(t))], \\ v_2(t) &= [(z_{i+1}(t) - z_i(t))(z_{i+1}(t) - z_i(t))] \times (z_{i+1}(t) - z_i(t)). \end{aligned}$$

Here is the main result of this paper.

Theorem 2.1. Let $y_0 \in H(\Omega)$; $T > 0$; $k_i > 0, i = 1, \dots, n-1$; $v_i \in L^\infty(0, T), i = 1, \dots, n-2$; and $z_i(0) \in \Omega, i = 1, \dots, n$, and let Assumptions 2.1 and 2.2 hold. Then there exists a $T^* = T^*(z_1(0), \dots, z_n(0), \|v_1\|_{L^\infty(0, T)}, \dots, \|v_{n-2}\|_{L^\infty(0, T)}, \Omega) \in (0, T)$ such that system (1.1) - (1.3) admits a unique solution $\{y, p, z\}$ on $(0, T^*)$, $\{y, \nabla p, z\} \in L^2(0, T^*; J_o(\Omega)) \times L^2(0, T^*; G(\Omega)) \times [C([0, T^*]; R^3)]^n$. Moreover, $y \in C([0, T^*]; H(\Omega))$, $y_t, y_{x_i x_j}, p_{x_i} \in (L^2(Q_{T^*}))^3$, where $i, j = 1, 2, 3$, and equations (1.1) and (1.2) are satisfied almost everywhere, while Assumptions 2.1 and 2.2 hold in $[0, T^*]$.

Remark 2.1.

- The fact that conditions Assumptions 2.1 and 2.2 hold in $[0, T^*]$ implies that we are able to guarantee that within $[0, T^*]$ no parts of the swimmer's body will "collide", and simultaneously, that it stays strictly inside of Ω . These conditions allow us to maintain the mathematical and physical wellposedness of model (1.1) - (1.3).
- As it will follow from the proof below, Theorem 2.1 allows further extension of the solutions to (1.1) - (1.3) in time as long as Assumptions 2.1 and 2.2 continue to hold. This depends on the choice of parameters $v_1(t), \dots, v_{n-2}(t)$.

Our plan to prove Theorem 2.1 is to proceed stepwise as follows:

- In Section 3 we discuss the existence and uniqueness of the solutions to the decoupled version of (1.2).
- In Section 4 we will introduce three continuous mappings for the decoupled version of the system (1.1) - (1.3).
- In Section 5 we will apply a fixed point argument to prove Theorem 2.1.

In the proofs below we employ the methods introduced in [19] to investigate the wellposedness of the 2- D version of model (1.1)-(1.3), modifying and extending them to the 3- D case.

Without loss of generality, we will further assume that system (1.1)-(1.3) and all respective auxiliary systems below are considered on the time-intervals whose lengths are smaller than 1.

3. Preliminary results. Introduce the following decoupled version of system (1.2):

$$\frac{dw_i}{dt} = \frac{1}{\text{mes}(S_i(0))} \int_{S_i(w_i(t))} u(x, t) dx, \quad w_i(0) = z_{i,0}, \quad i = 1, \dots, n, \quad (3.1)$$

where $u(x, t)$ is some given function. Denote $w(t) = (w_1(t), \dots, w_n(t))$.

Lemma 3.1. *Let $T > 0$ and $u \in (L^2(0, T; L^\infty(\Omega)))^3$ be given. Then there is a $T^* \in (0, T)$ such that system (3.1) has a unique solution in $C([0, T^*]; R^3)$ satisfying Assumptions 2.1 and 2.2 with $w(t)$ in place of $z(t)$, if they hold at time $t = 0$.*

Proof. We will use the contraction principle to prove existence and uniqueness. Below the values of h_0, C are taken from (2.2).

Select T_0 to satisfy the following inequalities:

$$\begin{aligned} 0 < T_0 &< \min \left\{ \frac{\text{mes}(S_0)h_0^2}{4\|u\|_{(L^2(Q_T))^3}^2}, \frac{(\text{mes}(S_0))^2}{C^2\|u\|_{(L^2(0,T;L^\infty(\Omega)))^3}^2} T, 1 \right\} \\ &\leq \min_{i=1,\dots,n} \min \left\{ \frac{\text{mes}(S_i(0))h_0^2}{4\|u\|_{(L^2(Q_T))^3}^2}, \frac{(\text{mes}(S_i(0)))^2}{C^2\|u\|_{(L^2(0,T;L^\infty(\Omega)))^3}^2} T, 1 \right\}. \end{aligned} \quad (3.2)$$

Let for any given $p \in C([0, T_0]; R^3)$:

$$B_{h_0/2}(p) = \left\{ z \in C([0, T_0]; R^3) \mid \|z - p\|_{C([0, T_0]; R^3)} \leq \frac{h_0}{2} \right\} \subset C([0, T_0]; R^3).$$

For each $i = 1, \dots, n$, define a mapping $D_i : B_{h_0/2}(z_{i,0}) \longrightarrow C([0, T_0]; R^3)$ by

$$D_i(w_i(t)) = z_{i,0} + \frac{1}{\text{mes}(S_i(0))} \int_0^t \int_{S_i(w_i(\tau))} u(x, \tau) dx d\tau.$$

Then we can derive that:

$$\|D_i(w_i(t))\|_{R^3} \leq \|z_{i,0}\|_{R^3} + \frac{\sqrt{T_0}}{\sqrt{\text{mes}(S_i(0))}} \|u\|_{(L^2(Q_T))^3} \quad \forall t \in [0, T_0], \quad i = 1, \dots, n. \quad (3.3a)$$

Similarly, in view of (3.2):

$$\|D_i(w_i(t)) - z_{i,0}\|_{C([0, T_0]; R^3)} \leq \frac{\sqrt{T_0}}{\sqrt{\text{mes}(S_i(0))}} \|u\|_{(L^2(Q_T))^3} < \frac{h_0}{2}. \quad (3.3b)$$

Thus, D_i maps $B_{h_0/2}(z_{i,0})$ into itself for each $i = 1, \dots, n$, where $z_{i,0}$ is treated as a constant function.

Let $w_i^{(1)}(t), w_i^{(2)}(t) \in B_{h_0/2}(z_{i,0})$ and $\xi(x, S)$ denote the characteristic function of a set $S \subset R^3$. Then, making use of (2.2), we obtain:

$$\begin{aligned} & \|D_i(w_i^{(1)}(t)) - D_i(w_i^{(2)}(t))\|_{R^3} = \\ & \frac{1}{\text{mes}(S_i(0))} \left\| \int_0^t \int_{S_i(w_i^{(1)}(\tau))} u(x, \tau) dx d\tau - \int_0^t \int_{S_i(w_i^{(2)}(\tau))} u(x, \tau) dx d\tau \right\|_{R^3} \\ & = \frac{1}{\text{mes}(S_i(0))} \left\| \int_0^t \int_{\Omega} u(x, \tau) \left(\xi(x, S_i(w_i^{(1)}(\tau))) - \xi(x, S_i(w_i^{(2)}(\tau))) \right) dx d\tau \right\|_{R^3} \\ & \leq \frac{C\sqrt{T_0}}{\text{mes}(S_i(0))} \|u\|_{(L^2(0,T;L^\infty(\Omega)))^3} \|w_i^{(1)}(t) - w_i^{(2)}(t)\|_{C([0,T_0];R^3)}, \quad \forall t \in [0, T_0], \quad i = 1, \dots, n. \end{aligned} \quad (3.4)$$

Therefore, after maximizing the left-hand side of (3.4) over $[0, T_0]$, we conclude:

$$\|D_i(w_i^{(1)}(t)) - D_i(w_i^{(2)}(t))\|_{C([0,T_0];R^3)} \leq \frac{C\sqrt{T_0}}{\text{mes}(S_i(0))} \|u\|_{(L^2(0,T;L^\infty(\Omega)))^3} \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T_0];R^3)}. \quad (3.5)$$

Hence, in view of (3.2),

$$\frac{C\sqrt{T_0}}{\text{mes}(S_i(0))} \|u\|_{(L^2(0,T;L^\infty(\Omega)))^3} < 1,$$

it follows from (3.5) that D_i is a contraction mapping on $B_{h_0/2}(z_{i,0})$ for each $i = 1, \dots, n$. Therefore, there exist unique $w_i(t) \in C([0, T_0]; R^3)$, $i = 1, \dots, n$, such that $D_i(w_i(t)) = w_i(t)$, i.e.,

$$w_i(t) = z_{i,0} + \frac{1}{\text{mes}(S_i(0))} \int_0^t \int_{S_i(w_i(\tau))} u(x, \tau) dx d\tau, \quad i = 1, \dots, n, \quad (3.6)$$

which yields (3.1).

On restrictions (2.3): Estimates (3.3a-b) imply that we may select a $T^* \in (0, T_0)$ such that for any $t \in [0, T^*]$, all $w_i(t)$'s will stay "close enough" to their initial values $z_{i,0}$'s to guarantee that Assumptions 2.1 and 2.2 holds for $w_i(t)$, $i = 1, \dots, n$. This ends the proof of Lemma 3.1.

4. Decoupled solution mappings. Let $\mathcal{B}_q(0)$ denote a closed ball of radius q (its value will be selected in Section 5) with center at the origin in the Banach space $L^2(0, T; J_o(\Omega)) \cap L^2(0, T; (H^2(\Omega))^3)$ endowed with the norm of $L^2(0, T; (H^2(\Omega))^3)$:

$$\mathcal{B}_q(0) = \left\{ \phi \in L^2(0, T; J_o(\Omega)) \cap L^2(0, T; (H^2(\Omega))^3) \mid \|\phi\|_{L^2(0,T;(H^2(\Omega))^3)} \leq q \right\},$$

where $H^2(\Omega) = \{\phi \mid \phi, \phi_{x_i}, \phi_{x_i x_j} \in L^2(\Omega), i, j = 1, 2, 3\}$.

Note that $H^2(\Omega)$ is continuously embedded into $C(\bar{\Omega})$, and thus $L^2(0, T; (H^2(\Omega))^3)$ is continuously embedded into $(L^2(0, T; L^\infty(\Omega)))^3$. This yields the estimate

$$\|\phi\|_{(L^2(0,T;L^\infty(\Omega)))^3} \leq K \|\phi\|_{L^2(0,T;(H^2(\Omega))^3)} \quad \text{for some } K > 0. \quad (4.1)$$

This implies that Lemma 3.1 holds for any $u \in \mathcal{B}_q(0)$.

4.1. Solution mapping for $z_i(t)$, $i = 1, \dots, n$. We now intend to show that the operator

$$\mathbf{A} : \mathcal{B}_q(0) \longrightarrow [C([0, T]; R^3)]^n, \quad \mathbf{A}u = w = (w_1, \dots, w_n),$$

where the w_i 's solve (3.1), is continuous and compact if $T > 0$ is sufficiently small.

Continuity. Let $u^{(1)}, u^{(2)} \in \mathcal{B}_q(0)$ with T_1 in place of T , where $T_1 > 0$ satisfies assumptions in the proof of Lemma 3.1 with T_1 in place of T^* . Define $\mathbf{A}u^{(j)} = w^{(j)} = (w_1^{(j)}, \dots, w_n^{(j)})$ for $j = 1, 2$. To show \mathbf{A} is continuous, we will evaluate

$$\|\mathbf{A}u^{(1)} - \mathbf{A}u^{(2)}\|_{[C([0,T];R^3)]^n}$$

term-by-term. To this end, similar to (3.4), we have the following estimate:

$$\begin{aligned} & \|w_i^{(1)}(t) - w_i^{(2)}(t)\|_{R^3} \\ &= \left\| \frac{1}{\text{mes}(S_i(0))} \int_0^t \int_{S(w_i^{(1)}(\tau))} u^{(1)}(x, \tau) dx d\tau - \frac{1}{\text{mes}(S_i(0))} \int_0^t \int_{S(w_i^{(2)}(\tau))} u^{(2)}(x, \tau) dx d\tau \right\|_{R^3} \\ &= \frac{1}{\text{mes}(S_i(0))} \left\| \int_0^t \int_{S(w_i^{(1)}(\tau))} u^{(1)}(x, \tau) dx d\tau - \int_0^t \int_{S(w_i^{(1)}(\tau))} u^{(2)}(x, \tau) dx d\tau \right. \\ &\quad \left. + \int_0^t \int_{S(w_i^{(1)}(\tau))} u^{(2)}(x, \tau) dx d\tau - \int_0^t \int_{S(w_i^{(2)}(\tau))} u^{(2)}(x, \tau) dx d\tau \right\|_{R^3} \\ &\leq \frac{1}{\text{mes}(S_i(0))} \left(\left\| \int_0^t \int_{\Omega} (u^{(1)}(x, \tau) - u^{(2)}(x, \tau)) \xi(x, S(w_i^{(1)}(\tau))) dx d\tau \right\|_{R^3} \right. \\ &\quad \left. + \left\| \int_0^t \int_{\Omega} u^{(2)}(x, \tau) (\xi(x, S(w_i^{(1)}(\tau))) - \xi(x, S(w_i^{(2)}(\tau)))) dx d\tau \right\|_{R^3} \right) \\ &\leq \frac{\sqrt{T_1}}{\sqrt{\text{mes}(S_i(0))}} \|u^{(1)} - u^{(2)}\|_{(L^2(Q_{T_1}))^3} + \frac{C\sqrt{T_1}}{\text{mes}(S_i(0))} \|u^{(2)}\|_{(L^2(0,T_1;L^\infty(\Omega)))^3} \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T_1];R^3} \\ &\leq \frac{\sqrt{T_1}}{\sqrt{\text{mes}(S_0)}} \|u^{(1)} - u^{(2)}\|_{(L^2(Q_{T_1}))^3} + \frac{C\sqrt{T_1}}{\text{mes}(S_0)} \|u^{(2)}\|_{(L^2(0,T_1;L^\infty(\Omega)))^3} \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T_1];R^3}. \quad (4.2) \end{aligned}$$

Recall from (4.1) that

$$\|u^{(2)}\|_{(L^2(0,T_1;L^\infty(\Omega)))^3} \leq Kq.$$

So, select a $T > 0$ as follows:

$$0 < T < \min \left\{ \left(\frac{\text{mes}(S_0)}{CKq} \right)^2, T_1 \right\}. \quad (4.3)$$

Hence, replacing T_1 in (4.2) with T satisfying (4.3) and maximizing the left-hand side of (4.2) over $[0, T]$, we obtain:

$$\begin{aligned} \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T];R^2)} &\leq \frac{\sqrt{T}}{\sqrt{\text{mes}(S_0)}} \|u^{(1)} - u^{(2)}\|_{(L^2(Q_T))^3} \\ &\quad + \frac{CKq\sqrt{T}}{\text{mes}(S_0)} \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T];R^3)}. \end{aligned}$$

In view of (4.3), if $w_i^{(1)}(t) \neq w_i^{(2)}(t)$ on $[0, T]$, then the above implies:

$$0 < \left(1 - \frac{CKq\sqrt{T}}{\text{mes}(S_0)} \right) \|w_i^{(1)} - w_i^{(2)}\|_{C([0,T];R^3)} \leq \frac{\sqrt{T}}{\sqrt{\text{mes}(S_0)}} \|u^{(1)} - u^{(2)}\|_{(L^2(Q_T))^3}.$$

Thus, it follows that

$$\|w_i^{(1)} - w_i^{(2)}\|_{C([0,T];R^3)} \leq \frac{\sqrt{T \text{mes}(S_0)}}{\text{mes}(S_0) - CKq\sqrt{T}} \|u^{(1)} - u^{(2)}\|_{(L^2(Q_T))^3}. \quad (4.4)$$

Therefore, (4.3) and (4.4) imply that for every $u^{(1)}, u^{(2)} \in \mathcal{B}_q(0)$,

$$\|\mathbf{A}u^{(1)} - \mathbf{A}u^{(2)}\|_{[C([0,T];R^3)]^n} \leq \frac{\sqrt{nT \text{mes}(S_0)}}{\text{mes}(S_0) - CKq\sqrt{T}} \|u^{(1)} - u^{(2)}\|_{L^2(0,T;(H^2(\Omega))^3)}.$$

So, the operator \mathbf{A} is continuous on $\mathcal{B}_q(0)$ for sufficiently small T as in (4.3).

Compactness. Furthermore, to show that \mathbf{A} is compact, we will show that \mathbf{A} maps any sequence in $\mathcal{B}_q(0)$ into a sequence in $[C([0,T];R^3)]^n$ which contains a convergent subsequence. To this end, consider any sequence $\{u^{(j)}\}_{j=1}^\infty$ in $\mathcal{B}_q(0)$. Using (3.6) with $w_i^{(j)}$ and $u^{(j)}$ in place of w_i and u , construct the sequence $\{w_i^{(j)}\}_{j=1}^\infty$, $i = 1, \dots, n$.

Let us now show that $\{w_i^{(j)}\}_{j=1}^\infty$ is uniformly bounded and equicontinuous. Indeed, applying (4.1) to an estimate like (3.3a) and then maximizing over $[0, T]$ yields:

$$\|w_i^{(j)}\|_{C([0,T];R^3)} \leq \max_{i=1,\dots,n} \{\|z_{i,0}\|_{R^3}\} + \frac{q\sqrt{T}}{\sqrt{\text{mes}(S_0)}}. \quad (4.5)$$

To show equicontinuity, consider any $t, t+h \in [0, T]$, e.g., when $h > 0$. Then for $i = 1, \dots, n$:

$$\begin{aligned} \|w_i^{(j)}(t+h) - w_i^{(j)}(t)\|_{R^3} &= \frac{1}{\text{mes}(S_i(0))} \left\| \int_t^{t+h} \int_{S(w_i^{(j)}(\tau))} u^{(j)}(x, \tau) dx d\tau \right\|_{R^3} \\ &\leq \frac{\sqrt{h}}{\sqrt{\text{mes}(S_i(0))}} \|u^{(j)}\|_{(L^2(Q_T))^3} \leq \frac{q\sqrt{h}}{\sqrt{\text{mes}(S_0)}} \quad j = 1, \dots, \end{aligned}$$

which implies the equicontinuity of $\{w_i^{(j)}\}_{j=1}^\infty$, $i = 1, \dots, n$ (the case $h < 0$ is similar). Therefore, by Ascoli's Theorem, $\{\mathbf{A}u^{(j)}\}_{j=1}^\infty$ contains a convergent subsequence in $[C([0,T];R^3)]^n$, i.e., \mathbf{A} is compact on $\mathcal{B}_q(0)$.

4.2. Solution mapping for decoupled Stokes equations. Now, consider the following decoupled Stokes initial boundary value problem:

$$\begin{aligned} \frac{\partial y_*}{\partial t} - \nu \Delta y_* + \nabla p_* &= f(x, t) \quad \text{in } Q_T, \\ \text{div } y_* &= 0 \quad \text{in } Q_T, \quad y_* = 0 \quad \text{in } \Sigma_T, \quad y_*|_{t=0} = y_0 \in H(\Omega). \end{aligned} \quad (4.6)$$

For any $f \in (L^2(Q_T))^3$, it is known that the boundary value problem (4.6) possesses a unique solution y_* in $L^2(0, T; J_o(\Omega)) \cap L^2(0, T; (H^2(\Omega))^3)$ with the properties described in Theorem 2.1 (see, e.g. [19], [27]). Moreover, (see, e.g. (7) on p. 79 and (50) on p. 65 in [19]), there is a positive constant L such that:

$$\|y_*\|_{L^2(0,T;(H^2(\Omega))^3)}^2 \leq L\|y_0\|_{H(\Omega)}^2 + L \int_{Q_T} \|f(\cdot, \tau)\|_{R^3}^2 d\tau. \quad (4.7)$$

Thus it follows that, given y_0 , the operator

$$\mathbf{B} : (L^2(Q_T))^3 \longrightarrow L^2(0, T; J_o(\Omega)) \bigcap L^2(0, T; (H^2(\Omega))^3), \quad \mathbf{B}f = y_*$$

is continuous.

4.3. The term $\mathbf{F}(z, v)$. Let $T > 0$ be as in (4.3). Given $u \in \mathcal{B}_q(0)$, let $F_*(w)$, $w = (w_1, \dots, w_n)$ denote the value of $F(z, v)$ in (1.3), with w_i 's from (3.6) in z_i 's, respectively. Consider the operator

$$\mathbf{F} : [C([0, T]; R^3)]^n \longrightarrow (L^2(Q_T))^3, \quad \mathbf{F}w = F_*(w).$$

We will show that \mathbf{F} is continuous, but first we evaluate $\|F_*(w)\|_{(L^2(Q_T))^2}$. To do that, we will use the standard algebraic transformation technique similar to that used for the 2- D swimming model in [19], [20]. Therefore, we omit some of the details below.

Let $P(T)$ denote the upper bound in (4.5):

$$P(T) = \max_{i=1, \dots, n} \{\|z_{i,0}\|_{R^3}\} + \frac{q\sqrt{T}}{\sqrt{\text{mes}(S_0)}}. \quad (4.8)$$

Then, using (4.5) with w_i in place of $w_i^{(j)}$, we can evaluate the first term in square brackets in the first line of (1.3) as follows:

$$\begin{aligned} & \left\| \xi_i(x, \tau) k_{i-1} \frac{(\|w_i(t) - w_{i-1}(t)\|_{R^3} - l_{i-1})}{\|w_i(t) - w_{i-1}(t)\|_{R^3}} (w_i(t) - w_{i-1}(t)) \right\|_{R^3} \\ & \leq C_o \left(n, r, P(T), \max_{i=1, \dots, n-1} \{k_i\}, \max_{i=1, \dots, n-1} \{l_i\} \right), \end{aligned}$$

where and below $C_0 = C_0(n, r, P(T), \max_{i=1, \dots, n-1} \{k_i\}, \max_{i=1, \dots, n-1} \{l_i\})$ denotes a positive generic constant, continuous in r , k_i 's, l_i 's, and $P(T)$ (it can have a different concrete value for different expressions below). Along similar estimates for other terms in (1.3), we obtain:

$$\|F_*(w)\|_{(L^2(Q_T))^3} \leq C_o \sqrt{T \text{mes}(\Omega)} \left(1 + \max_{i=1, \dots, n-2} \{\|v_i\|_{L^\infty(0, T)}\} \right) \quad \forall u \in \mathcal{B}_q(0). \quad (4.9)$$

We will now show that \mathbf{F} is a continuous operator. Let $w^{(1)}, w^{(2)} \in [C([0, T]; R^3)]^n$. Consider (1.3) with $w_i^{(j)}$ in place of z_i , $i = 1, \dots, n$, $j = 1, 2$.

Making use of estimate like (3.3b), without loss of generality we can assume that T is small enough to ensure that Assumptions 2.1 and 2.2 holds (as stated in Lemma 3.1). Then, once again, similar to the 2- D case model in [19], we can evaluate $\|F_*(w^{(1)}) - F_*(w^{(2)})\|_{R^3}$ along the standard algebraic transformations and making use of Assumption 2.2, which will result in the following formula:

$$\|F_*(w^{(1)}) - F_*(w^{(2)})\|_{(L^2(Q_T))^3} \leq M(T) \sqrt{T \text{mes}(\Omega)} \|w^{(1)} - w^{(2)}\|_{[C([0, T]; R^3)]^n}, \quad (4.10)$$

where

$$M(T) = M(T, n, r, P(T), \max_{i=1, \dots, n-1} \{k_i\}, \max_{i=1, \dots, n-1} \{l_i\}) \left(1 + \max_{j=1, \dots, n-2} \{\|v_j\|_{L^\infty(0, T)}\} \right), \quad (4.11)$$

is defined similar to C_o . Hence \mathbf{F} is a continuous operator for T satisfying (4.3).

5. Proof of Theorem 2.1. Before proceeding to the proof of Theorem 2.1, we summarize the main results of the previous section. In Section 4, we proved that for sufficiently small $T > 0$ satisfying (4.3), the operators

$$\mathbf{A} : \mathcal{B}_q(0) \longrightarrow [C([0, T]; R^3)]^n, \quad \mathbf{A}u = w = (w_1, \dots, w_n),$$

$$\mathbf{F} : [C([0, T]; R^3)]^n \longrightarrow (L^2(Q_T))^3, \quad \mathbf{F}w = F_*(w),$$

and

$$\mathbf{B} : (L^2(Q_T))^3 \longrightarrow L^3(0, T; J_o(\Omega)) \bigcap L^2(0, T; (H^2(\Omega))^3), \quad \mathbf{B}f = y_*$$

are all continuous. \mathbf{A} is also compact. As a result,

$$\mathbf{BFA} : \mathcal{B}_q(0) \longrightarrow L^2(0, T; J_o(\Omega)) \bigcap L^2(0, T; (H^2(\Omega))^3), \quad \mathbf{BFA}u = y_*$$

is continuous and compact.

5.1. Existence: Fixed point argument. Select the value of q to be any positive number larger than $\sqrt{L}\|y_0\|_{H(\Omega)}$, and choose $T > 0$ as in (4.3) so that Lemma 3.1 holds. In view of (4.7) (with F_* in place of f), we select $T^* \in (0, \min\{T, 1\})$ small enough so that the continuous and compact operator \mathbf{BFA} maps the closed ball $\mathcal{B}_q(0)$ into itself.

Re-write (4.9) as follows:

$$\|F_*(w)\|_{(L^2(Q_T))^3} < C_1\sqrt{T},$$

where the positive constant C_1 is independent of T . Select T^* so that:

$$0 < T^* < \min \left\{ \frac{q^2 - L\|y_0\|_{H(\Omega)}^2}{LC_1^2}, T, 1 \right\}. \quad (5.1)$$

Since T^* satisfies (4.3), if we replace T with T^* in (4.7), (4.9) - (4.11), we obtain:

$$\begin{aligned} \|\mathbf{BFA}u\|_{L^2(0, T^*; (H^2(\Omega))^3)}^2 &\leq L\|y_0\|_{H(\Omega)}^2 + L\|F_*(w)\|_{(L^2(Q_{T^*}))^3}^2 \\ &< L\|y_0\|_{H(\Omega)}^2 + LC_1^2 T^* < L\|y_0\|_{H(\Omega)}^2 + q^2 - L\|y_0\|_{H(\Omega)}^2 = q^2. \end{aligned}$$

Hence, \mathbf{BFA} maps $\mathcal{B}_q(0)$ into itself if (5.1) is satisfied.

Thus, by Schauder's Fixed Point Theorem, \mathbf{BFA} has a fixed point y which is a solution of the system (1.1) - (1.3), and which satisfies all of the requirements of Theorem 2.1. As usual, we may select ∇p in $L^2(0, T^*; G(\Omega))$ to complement the solution $y \in L^2(0, T^*; J_o(\Omega))$ in Theorem 2.1. This completes the proof of existence for Theorem 2.1.

5.2. Uniqueness. To prove that the solution found in Section 5.1 is unique, we will argue by contradiction. Namely, suppose, e.g., that there are two different solutions

$$\left\{ z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)}), y^{(1)}, p^{(1)} \right\}$$

and

$$\left\{ z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)}), y^{(2)}, p^{(2)} \right\}$$

to (1.1)-(1.3), satisfying the properties described in Theorem 2.1 on some time interval $[0, T]$, where T satisfies inequality (5.1). Without loss of generality, we assume the two solutions are different right from $t = 0$.

By (4.4), with $z_i^{(j)}$ in place of $w_i^{(j)}$, $i = 1, \dots, n$, and $y^{(j)}$ in place of $u^{(j)}$, both for $j = 1, 2$, we see that for any $T_0 \in (0, T]$, and each $i = 1, \dots, n$:

$$\|z_i^{(1)} - z_i^{(2)}\|_{C([0, T_0]; R^3)} \leq \frac{\sqrt{T_0 \text{mes}(S_0)}}{\text{mes}(S_0) - CKq\sqrt{T_0}} \|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3}. \quad (5.2)$$

Let us now evaluate $\|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3}$. Note that $(y^{(1)} - y^{(2)})$ satisfies the following Stokes initial-value problem:

$$\begin{aligned} \frac{\partial(y^{(1)} - y^{(2)})}{\partial t} &= \nu \Delta(y^{(1)} - y^{(2)}) + (F(z^{(1)}, v) - F(z^{(2)}, v)) - \nabla(p^{(1)} - p^{(2)}) \quad \text{in } Q_{T_0}, \\ \text{div}(y^{(1)} - y^{(2)}) &= 0 \quad \text{in } Q_{T_0}, \quad (y^{(1)} - y^{(2)}) = 0 \quad \text{in } \Sigma_{T_0}, \quad (y^{(1)} - y^{(2)})|_{t=0} = 0. \end{aligned}$$

According to (4.7) we have:

$$\|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3}^2 \leq L \int_0^{T_0} \int_{\Omega} \|F(z^{(1)}, v) - F(z^{(2)}, v)\|_{R^3}^2 dx dt. \quad (5.3)$$

In turn, similar to (4.10):

$$\|F(z^{(1)}, v) - F(z^{(2)}, v)\|_{R^3} \leq N(T) \sum_{j=1}^n \|z_j^{(1)} - z_j^{(2)}\|_{C([0, T_0]; R^3)}. \quad (5.4)$$

for some $N(T)$ is nonincreasing at $T \rightarrow 0^+$. Hence, combining (5.2) - (5.4) yields:

$$\|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3} \leq \frac{nN(T)T_0\sqrt{L\text{mes}(S_0)\text{mes}(\Omega)}}{\text{mes}(S(0)) - CKq\sqrt{T_0}} \|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3}. \quad (5.5)$$

Now, select T_0 as follows:

$$0 < T_0 < \min \left\{ \frac{\text{mes}(S_0)}{4nN(T)\sqrt{L\text{mes}(S_0)\text{mes}(\Omega)}}, \left(\frac{\text{mes}(S_0)}{2CKq} \right)^2, T \right\}. \quad (5.6)$$

This choice of T_0 implies that the following inequality holds:

$$nN(T)T_0\sqrt{L\text{mes}(S_0)\text{mes}(\Omega)} + \frac{1}{2}CKq\sqrt{T_0} < \frac{1}{2}\text{mes}(S_0).$$

So, it follows from (5.5), (5.6) that:

$$\|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3} < \frac{1}{2} \|y^{(1)} - y^{(2)}\|_{(L^2(Q_{T_0}))^3}.$$

Therefore $y^{(1)} \equiv y^{(2)}$ on $[0, T_0]$, and thus by (5.2), $z_i^{(1)} \equiv z_i^{(2)}$ for $i = 1, \dots, n$ on $[0, T_0]$. Contradiction. This ends the proof of Theorem 2.1.

6. Conclusion. In this paper we introduced a new hybrid model describing the locomotion of a “small” swimmer in the incompressible 3-*D* fluid. The model consists of two coupled systems of equations: one is a PDE system— for the fluid, governed by the nonstationary 3-*D* Stokes equation (typically associated with the low Reynolds numbers), and the other is an ODE system— for the position of swimmer in it. It is assumed that the swimmer’s body consists of finitely many subsequently connected parts, identified with the fluid they occupy, which in turned are linked by the rotational and elastic Hooke’s forces. We investigated the well-posedness of this model. Namely, in suitable function spaces we obtained the existence, uniqueness and regularity results for its solutions. These results are to be used for the derivation of a formula for asymptotically small motions of this type of 3-*D* swimmers, see [21]-[22] (and [20] for the 2-*D* case). They can also be instrumental to study controllability properties of this swimming model [20]. Models like this are of interest in biological and engineering applications dealing with the study and design of self-propulsion systems in fluids.

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